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INDUCING, SLOPES, AND CONJUGACY CLASSES

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ABSTRACT

We show that the conjugacy class of an eventually expanding continuous piecewise affine interval map is contained in a codimension 1 submanifold of parameter space. In particular conjugacy classes have empty interior. This is based on a study of the relation between induced Markov maps and ergodic theoretical behavior.

1. Introduction

One of the central questions in iteration theory is to decide whether two maps $f: X \to X$ and $g: Y \to Y$ are topologically conjugate, i.e. whether there exists a homeomorphism $h: X \to Y$ such that $h \circ f = g \circ h$. In this paper we deal with this question for eventually expanding piecewise affine maps on the interval. A

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map is called **piecewise affine** if it is continuous, piecewise monotone, and affine on each of its finitely many intervals of monotonicity.

The unimodal case was already studied by Misiurewicz and Visinescou (see [MiV] which also refers to former literature). They showed that the conjugacy classes form lines in the two parameter family of the eventually expanding unimodal piecewise affine maps. In this paper we will show that the conjugacy class of an eventually expanding piecewise affine map is contained in a codimension 1 submanifold of parameter space.

The following trivial remark is the key to our study of the conjugacy classes of piecewise affine maps. Consider a set A, finite or denumerable, and assume that the interval I is, up to a set of measure zero, the pairwise disjoint union of intervals I_a, $a \in A$. Then a map $f\colon \bigcup I_a \to I$, where f carries each I_a onto I by an affine homeomorphism, is called a **multiple covering map (with index set** A**)**: notice that the domain of such a map has full measure in its image (in short, **has full measure**). The derivative of the branch $f|I_a$ is denoted by Df_a.

Multiple Covering Map Principle: *Let f be a multiple-covering map with index set A. Then*

$$\Sigma_{a\in A}\frac{1}{|Df_a|} = 1.$$

In particular, for multiple covering maps f, g which have the same index set A:

$$\{\forall a \in A,\ |Df_a| \geq |Dg_a|\} \ \Rightarrow\ \{\forall a \in A,\ |Df_a| = |Dg_a|\}\,.$$

For piecewise affine Markov maps, the above principle applies almost immediately, yielding similar results. As we shall see, a much finer analysis is required to deal with more general piecewise affine maps. The first step of our study is to associate induced Markov maps (see Section 4) to piecewise affine maps. These induced maps have a topological definition and look like multiple covering maps. The only difference is that we don't know whether, for such a map, the domain of definition has full measure. So, before applying the multiple covering map principle to induced Markov maps, we have to study the measure of their domain of definition.

A piecewise affine map has the **Markov property** if it has an induced Markov map whose domain of definition has full measure, i.e., it is a multiple covering maps (see Section 4). A closed set $A \subset N$ is called an **absorbing** set of the interval map $f\colon N \to N$ if

$$\{x \in N | \omega(x) \subset A\}$$

has positive Lebesgue measure (where $\omega(x)$ denotes the positive f-limit set of $x \in N$). In [M] it was shown that S-unimodal maps have the Markov property if and only if the map does not have zero-dimensional absorbing sets. The main part of this paper is devoted to proving the same result for piecewise affine maps. In Section 4 we prove

THEOREM A: *A piecewise affine map has the Markov property if and only if it does not have zero-dimensional absorbing sets.*

There are three different properties which allow a map to have a zero-dimensional absorbing set. A map can have a periodic attractor. Secondly it can be infinitely renormalizable (see Section 2). In this case the topological structure causes an absorbing Cantor set. Furthermore a non-renormalizable map can have an absorbing Cantor set, which is caused by intrinsic geometrical properties.

In [LM] and [L] it was shown that quadratic unimodal maps can only have an absorbing Cantor set if they are infinitely renormalizable. However, recently it has been shown in [BKNS] that there exist unimodal maps with highly degenerate critical point having absorbing Cantor sets. These results depend on a fine control of the geometry. We will avoid such geometrical studies by an ergodic theoretical shortcut: we only study eventually expanding piecewise affine maps, and for such maps, one knows the existence of absolutely continuous invariant probability measures [LY]. Yet, these measures cannot coexist with zero-dimensional absorbing sets and we get

THEOREM B: *Eventually expanding piecewise affine maps have the Markov property.*

However the study of the intrinsic geometry is just postponed. To prove the following conjecture one would probably have to go into geometrical considerations.

CONJECTURE: *A piecewise affine interval map with no periodic attractor is eventually expanding.*

In Section 5 we will apply the multiple covering map principle to the Markov maps which are now, by Theorem B, multiple covering maps. A **branch** of a map is the restriction to an interval of monotonicity. Branches of a piecewise affine map which contain pieces of the non-wandering set in the interior of its domain are called **essential branches.** Let $\mathcal{E}_d$ be the family of d-modal eventually expanding piecewise affine maps. $\mathcal{E}_d$ is naturally parametrized by some submanifold of $\mathbb{R}^{2d+2}$, to which we identify it. Studying the conjugacy problem in $\mathcal{E}_d$ we got the following result.

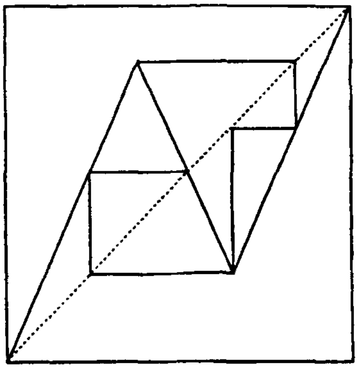

Figure 1.

THEOREM C: *Every conjugacy class is contained in a codimension 1 submanifold of $\mathcal{E}_d$. Furthermore, if the slope of some essential branches of $f \in \mathcal{E}_d$ are increased, the topological type changes.*

In the unimodal case we know that by moving up the critical value we will increase monotonically the kneading sequences and the entropy. In the multimodal case we also expected to change the topological type by moving up a critical value. Now this is proved to be true in Theorem C. However, in the multimodal case monotonicity questions relating kneading information and the study of isentrops, the level sets of entropy, are much more delicate. For example, moving up the value of the left critical point of the map given in Figure 1 will not increase both kneading sequences. Moving up the left critical value will decrease the left kneading sequence and increase the right kneading sequence. It is very well possible that moving up does not give rise to an increase of entropy.

Section 2 contains some basic topological lemmas, some of which are part of the folklore. To simplify the exposition of the proofs, we only considered maps on the interval: most of this paper, and in particular Theorems A, B and C, hold true as well for piecewise affine circle maps which have at least one periodic orbit. In Section 3 we define **good** intervals and describe their properties: they are the main ingredient in the definition of the Markov maps.

The proof of the characterization of maps with the Markov property, presented in Section 4, can easily be generalized by using the tools from [M]. Thus Theorem

A also holds for smooth multimodal maps with negative Schwarzian derivative. More work would be needed to get a similar C^2 result.

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QUESTION: *Do isentrops of eventually expanding piecewise affine maps have empty interior?*

NOTATIONS. We will use the following conventions and notations. Intervals will always have positive length. Let $N = [a, b]$ be an interval and $A, U \subset N$, where A is measurable and U open. Let $f: N \to N$ be a piecewise affine map.

∂U is the boundary of U,
$\text{int}(A)$ is the interior of A,
$\text{mesh}(U)$ is the length of the longest connected component of U,
$|A|$ is the Lebesgue measure of A,
C_f is the set of critical points of f,
$\text{orb}(A) = \{A, f(A), f^2(A), \dots\}$ is the orbit of A,
Df_i is the derivative of the i^{th} branch.

We shall also say that $U \subset N$ **satisfies the ∂-condition** if $\text{orb}(\partial U) \cap U = \emptyset$. A **branch** of a piecewise monotone map is the restriction of the map to a maximal interval on which it is monotone.

2. Non-renormalizable maps

Let us begin with some definitions.

The continuous map $f: N \to N$ is **piecewise affine** if there exist points $a = a_0 < a_1 < \dots < a_d < a_{d+1} = b$ such that $f_i = f|[a_i, a_{i+1}]$ is affine, and $Df_i Df_{i+1} < 0$. The points $a_1, a_2, \dots, a_d$ are called **critical points.** We say the map is **d-modal** when we want to stress the number of its critical points, **multimodal** when $d \geq 1$ and **unimodal** when $d = 1$.

Consider a piecewise affine map $f: N \to N$. With $I \subset N$ an interval and $n \geq 1$, the pair (I, n) is called a **renormalization** of f if

$f^n(I) \subset I$ and $\overline{I} \neq N$,
the interiors of $f^i(I)$, $i = 0, \dots, n-1$ are pairwise disjoint.

A map which has a renormalization is called **renormalizable.** The orbit $\bigcup f^i(I)$ is called a **cycle (with period n).** A cycle is called **minimal** if $f^n|I$

is non-renormalizable. A pair (I,n) is called a **trap** of the piecewise affine map $f\colon N \to N$ if $f^n(I) \subset I$ and $\overline{I} \neq N$.

The following two properties of non-renormalizable maps will be used over and over again.

LEMMA 2.1: *Let $f\colon N \to N$ be non-renormalizable and piecewise affine. Then*

(1) $\partial N \subset \mathrm{orb}(C_f)$,

(2) $f^{-1}(x) \setminus (\partial N \cup C_f) \neq \emptyset$ *for every* $x \notin \partial N$,

(3) *for every interval* $I \subset N$

$$\bigcup_{i\geq 0} f^i(I) = N.$$

Proof: The proof of (1) and (2) is easily supplied and we proceed with the proof of (3).

Let $I \subset N$ be an interval. Observe that f cannot have periodic attractors. In [MMS] it was proved that a piecewise affine map f without periodic attractor cannot contract intervals too much: $\inf_{i\geq 0} |f^i(I)| > 0$. This implies that the connected components of $\bigcup_{i\geq 0} f^i(I)$ have a definite size. Hence the set can have only a finite number of connected components. These components are permuted by f. In particular they are eventually periodic. This gives rise to a renormalization. Hence there is exactly one component which is dense in N. Clearly this invariant component contains C_f. Hence it contains $\mathrm{orb}(C_f) \supset \partial N$, so it is N. ∎ (Lemma 2.1)

LEMMA 2.2: *A non-renormalizable piecewise affine map does not have traps.*

Proof: Let $f\colon N \to N$ be a non-renormalizable piecewise affine map. Observe that every non-renormalizable map has at least one expanding fixed point in $\mathrm{int}(N) \setminus C_f$, say $f(p) = p$ and $|Df(p)| > 1$.

Assume that there is some trap (I,n). By Lemma 2.1 $p \in I$.

Assume $p \in \mathrm{int}(I)$ or $p \in \partial I$ is order preserving. Let

$$E_k = \{x \in I | \{x, f(x), \ldots, f^{k-1}(x) \subset I \text{ and } f^k(x) \notin I\}.$$

Because there are no renormalizations $f(I) \not\subset I$. Hence $E_1 \neq \emptyset$. Assume $E_k \neq \emptyset$ for some $k \geq 1$. We set $F_k = I \setminus (E_1 \cup E_2 \cup \cdots \cup E_k)$. From the assumption on p, F_k contains an interval, hence by Lemma 2.1 $f(F_k) \not\subset F_k$. By definition $f(F_k) \subset I$, thus $f^{-1}(E_k) \cap I = E_{k+1} \neq \emptyset$.

Now consider E_n: $E_n \neq \emptyset$ and $f^n(E_n) \cap I = \emptyset$. But I is a trap, so $f^n(E_n) \subset f^n(I) \subset I$, a contradiction.

Assume $p \in \partial I$ is an order reversing fixed point. Then $p \notin \mathrm{int}(f^i(I))$ for $i \geq 0$. Now $I \cup f(I)$ is also a trap but containing p in its interior and we are back to the previous case, which implies $I \cup f(I) \supset \mathrm{int}(N)$. But now, I and $f(I)$ are the components of $N - \{p\}$ and f interchanges these two components. We found a renormalization, a contradiction. ∎ (Lemma 2.2)

LEMMA 2.3: *The periodic points of a non-renormalizable piecewise affine map $f: N \to N$ are dense.*

Proof: Fix an open interval $I \subset N$ with $\overline{I} \cap \partial N = \emptyset$. The aim is to show that there exists $k \geq 1$ such that $f^k(I) \supset I$: I contains a periodic point.

Because the orbit of I is dense there exists a $q \geq 1$ such that $f^q(I) \cap I \neq \emptyset$. Let $I_j = f^{jq}(I)$, $j \geq 0$. Assume that $I \subset I_j$ never happens.

Consider $T_k = \bigcup_{j \leq k} I_j$, $k \geq 0$. Because $I_0 \cap I_1 \neq \emptyset$ every T_k is an interval. Clearly $f^q(T_k) \subset T_{k+1}$ and $T_k \subset T_{k+1}$. Let $I = (a, b)$ and suppose $a \notin I_1$.

CLAIM: *$a \notin T_k$ for $k \geq 0$.*

Proof of Claim: For $k = 0, 1$ the claim is true. Assume by contradiction that there exists a first $k \geq 1$ such that $a \in T_{k+1}$. Because f is non-renormalizable, by Lemma 2.2 we get that $T_l - T_{l-1} = J_l \neq \emptyset$ for $l \leq k$. Observe that $J_l \subset I_l$, hence J_l is an interval, otherwise $a \in T_l$. Now there exists $x \in \overline{J_{k-1}}$ with $f^q(x) = b_k$ where b_k is the right boundary point of T_k. Let $D = [x, b_{k-1}]$. Then $D \subset J_{k-1} \subset I_{k-1}$.

We have $a \in f^q(J_k)$ and $f^q(x) = b_k$, hence $T_k \subset f^q(D \cup J_k)$. Now $I_0 \not\subset f^q(J_k)$ by assumption, hence

$$D \cup J_k \subset T_k \setminus I_0 \subset f^q(D).$$

This yields

$$T_k \subset f^q(J_k \cup D) \subset f^{2q}(D),$$

hence

$$I_0 \subset f^{2q}(I_{k-1}) = I_{k+1},$$

a contradiction. ∎ (Claim)

To finish the proof of Lemma 2.3, let $T = \bigcup T_k$. Now $f^q(T) \subset T$ and the closure of T is not the whole N because T lies on one side of I and I does not touch the boundary of N. We found a trap, a contradiction. Hence for some $j \geq 1$ we have $I \subset I_j$. ∎ (Lemma 2.3)

Remember that an interval satisfies the ∂-condition if $\mathrm{orb}(\partial U) \cap U = \emptyset$.

COROLLARY 2.4: *The critical set of a non-renormalizable piecewise affine map has a neighborhood U satisfying the ∂-condition and having arbitrarily small mesh.*

LEMMA 2.5: *Let $c \in C_f$ be a critical point of the non-renormalizable piecewise affine map $f\colon N \to N$. Then in every component $M \subset N \setminus (\partial N \cup C_f)$ there exists an open interval $I \subset M$ and $n \geq 1$ such that $f^n|I$ is monotone and $c \in f^n(I)$.*

Proof: Fix $c \in C_f$. Let $d \in C_f$. If $c \in \mathrm{orb}(d)$ then $n_d \geq 0$ will stand for the first time that d hits c. Now take $L > \max\{n_d | n_d < \infty\}$.

Using Lemma 2.1 (2) we can choose a sequence $c_0 = c, c_{-1}, c_{-2}, \dots$ such that $f(c_{-(i+1)}) = c_{-i}$ and $c_{-i} \notin C_f \cup \partial N$. Then consider c_{-L}. By Lemma 2.1 (3) there is some $n \geq 1$ with $c_{-L} \in f^n(M)$. In particular there exists an interval $J_1 \subset M$ such that $f^n|J_1$ is monotone and $c_{-L} \in f^n(J_1)$. Choose J_1 to be maximal, which means that $f^n|J_1$ is a branch, i.e., by Lemma 2.1 (1) we know that $\partial f^n(J_1) \subset \mathrm{orb}(C_f)$.

Assume that $c_{-L} \in \partial f^n(J_1)$: then there would be some critical point $d \in C_f$ and some $i \geq 1$ such that $f^i(d) = c_{-L}$. Take the pair (d, i) with i minimal. Then $n_d = L + i > L$, contradicting the definition of L. Hence $c_{-L} \in \mathrm{int}(f^n(J_1))$.

To finish the proof, consider the orbit of c_{-L}. It does not pass trough critical points. Hence there is some open interval $J_2 \ni c_{-L}$ with $f^L|J_2$ monotone and $c \in f^L(J_2)$. And we can take $I = f^{-n}(J_2) \cap J_1$. ∎ (Lemma 2.5)

LEMMA 2.6: *Let $f\colon N \to N$ be a non-renormalizable piecewise affine map. Then for every interval I there exist an interval $J \subset I$ and $n \geq 1$ such that $f^n|J$ is monotone and $f^n(J)$ is a connected component of $N \setminus C_f$.*

Proof: Consider the interval $I \subset N$.

CLAIM: There exists $n \geq 0$ such that $f^n(I)$ contains a component of $N \setminus C_f$.

Proof of Claim: Because f does not have wandering intervals and periodic attractors, there exist infinitely many $n \geq 0$ with $f^n(I) \cap C_f \neq \emptyset$. In particular there exist a critical point $c \in C_f$ and non-negative numbers n, q such that $f^n(I) \ni c$ and $f^{n+q}(I) \ni c$. Now consider the interval $T = \bigcup_{j \geq 0} f^{jq}(f^{n+1}(I))$.

If either $f^n(I)$ or $f^{n+q}(I)$ contains two consecutive critical points, we are done. Otherwise, because f does not have traps, $f^{n+1}(I) \subset f^q(f^{n+1}(I))$ which implies

$$f^{jq}(f^{n+1}(I)) = \bigcup_{i\le j} f^{iq}(f^{n+1}(I)).$$

But $f^q(T) \subset T$ and, since f does not have traps, we get $\overline{T} = N$. By Lemma 2.1 (3) we have in fact $T = N$. Hence there exists $j_0 \ge 1$ such that $f^{j_0}(I) = \bigcup_{i\le j} f^{iq}(f^{n+1}(I)) = N$. ∎ (Claim)

We are going to prove that I contains an interval which is mapped after some time monotonically onto a component of $N \setminus C_f$. Assume $f^j(I)$ does not cover a component of $N \setminus C_f$ for $j < n$ but $f^n(I)$ does contain a component. For every $k \le n$ there exist $J_k \subset I$ such that

$f^k|J_k$ is monotone,

$f^k(J_k) = f^k(I)$.

To prove this let $J_0 = I$ and assume that J_k is defined for some $k < n$. If $f|f^k(I)$ is monotone then let $J_{k+1} = J_k$. If $f|f^k(I)$ is not monotone then there exists a unique critical point $c \in C_f$ with $c \in f^k(J_k) = f^k(I)$. Let $L, R \subset J_k$ be the intervals which are mapped onto the two components of $f^k(J_k) \setminus \{c\}$. We may assume that $f^{k+1}(L) \subset f^{k+1}(R)$. Now $f^{k+1}(I) = f^{k+1}(R)$ and $f^{k+1}|R$ is monotone. So let $J_{k+1} = R$.

To finish the proof of Lemma 2.6, we just choose $J = J_n$. ∎ (Lemma 2.6)

A piecewise affine map f is called **eventually expanding** if there is an integer $n \ge 1$ so that $|Df^n| > 1$ whenever this derivative is defined.

LEMMA 2.7: *Every eventually expanding piecewise affine map is non-renormalizable or has finitely many minimal cycles.*

Proof: Every cycle (I, n) consists of pairwise disjoint intervals. This implies that the number of critical points of renormalizations is uniformly bounded. Hence there is always a branch of $f^n|I$, say $f^n\colon J \to I$ monotone and $J \subset I$ whose size is a definite fraction of I. But $|Df^m| \to \infty$ so that f^n could not map this piece into I for n big. We conclude that the period of the renormalizations is bounded for eventually expanding maps. Since each cycle contains at least one critical point we conclude that there are only finitely many minimal cycles.

To find a minimal cycle take a renormalization with maximal period; (I, n). Then there will be a smallest interval $J \subset I$ such that (J, n) is still a renormalization. The orbit of J is a minimal cycle. ∎ (Lemma 2.7)

Remarks: (1) The interiors of minimal cycles are pairwise disjoint,

(2) Almost every point enters after some time a minimal cycle. In particular, every minimal cycle equals the conservative part of some ergodic component.

(3) All statements 2.1–2.7 remain true if piecewise affine is replaced by "continuous and with no homterval".

3. Good intervals

Fix a non-renormalizable piecewise affine map $f: N \to N$. An open set $U \supset C_f$ is called a **nice neighborhood** of C_f if it satisfies the ∂-condition and every connected component contains exactly one critical point. We set $U = \bigcup_{c \in C_f} U_c$. Corollary 2.4 states that there are nice neighborhoods U with $\text{mesh}(U)$ arbitrarily small.

Definition 3.1: Let $U_c \subset U \subset N$ be a component of the nice neighborhood U of C_f. An interval $T \subset N$ is called a **good interval (of time $n \geq 0$) for U_c** if $f^n: T \to U_c$ is monotone and onto.

Because every component of a nice neighborhood of C_f contains a critical point, every good interval has a well defined time needed for reaching the nice neighborhood. The ∂-condition implies easily that two intersecting good intervals T_1 and T_2 corresponding to the same nice neighborhood are nested: if $T_1 \cap T_2 \neq \emptyset$ then either $T_1 \subset T_2$ or $T_2 \subset T_1$. The following Lemma states that the collection of good intervals is big.

LEMMA 3.2: *Let $U \subset N$ be a nice neighborhood of C_f with* $\text{mesh}(U)$ *small enough. For every critical point $c \in C_f$ there exists, in every interval $I \subset N$, a good interval $T \subset I$ for U_c, whose time is at least* 1.

Proof: Fix $c \in C_f$. Lemma 2.5 says that in every component M of $N \setminus C_f$ there exists an open interval $I_M(c) \subset M$ and $n_M(c) \geq 1$ such that $c \in f^{n_M(c)}(I_M(c))$ and $f^{n_M(c)}|I_M(c)$ is monotone. Choose such an interval in every component M of $N \setminus C_f$. Let $V_c = \bigcap f^{n_M(c)}(I_M(c))$. By Lemma 2.6 we will find, in every interval $I \subset N$, an interval $J \subset I$ and $n \geq 1$ such that f^n: $J \to V_c$ is monotone and onto. Now Lemma 3.2 holds if we take U small enough such that $U \subset \bigcap V_c$. ∎ (Lemma 3.2)

Observe that we can describe topologically how small U has to be to apply Lemma 3.2. In Section 5 we will discuss conjugacy classes. For this we prefer to deal with topologically defined objects.

To avoid the annoying fact that the branches can be restricted by the boundary points of N, we assume that the map f is part of an **extension**, i.e. there is a piecewise affine map g: $[-1,1] \to [-1,1]$ such that

$N \subset [-1,1]$,
$g|N = f$,
$g(\{-1,1\}) \subset \{-1,1\}$,
every point in $(-1,1)$ enters N after some time.

The next Lemma explains why nice neighborhoods are nice.

LEMMA 3.3: *Let $U \supset C_f$ be a nice neighborhood. If $f^i(x) \notin U$ for $i < n$ but $f^n(x) \in U_c \subset U$, there exists a good interval $T \ni x$ of time n for U_c.*

Proof: Let $U = \bigcup_{c \in C_f} U_c$ be a nice neighborhood of C_f. Take $x \in N$ such that $f^i(x) \notin U$ for $i = 0, \ldots, n-1$ and $f^n(x) \in U_c$. Suppose by contradiction that $f^n(T)$ does not cover U_c, where $T \ni x$ is the maximal interval on which f^n is monotone. We assumed f to be part of an extension. Hence the monotonicity is restricted by some critical point: there exist $i \leq n-1$ and a critical point $d \in C_f$ such that $d \in \partial f^i(T)$ and $f^{n-i}((d, f^i(x)) \subset U_c$. By definition of n we know that $f^i(x) \notin U_d$. Hence $(d, f^i(x)) \cap \partial U_d \neq \emptyset$ which implies that $\mathrm{orb}(\partial U_d) \cap U \neq \emptyset$, a contradiction. ∎ (Lemma 3.3)

LEMMA 3.4: *Let U be a nice neighborhood for C_f. There exists a closed set Λ_U with Lebesgue measure zero such that every component of the complement of $\Lambda_U \cup U$ is a good interval for U whose time is at least* 1.

Proof: Let $U = \bigcup_{c \in C_f} U_c$ be a nice neighborhood of C_f and $\Lambda_U = \{x \in N | \mathrm{orb}(x) \cap U = \emptyset\}$. Choose a component S of the complement of Λ_U and $x \in S$. Assume that the orbit of x enters U for the first time in n steps, say $f^n(x) \in U_c$. This means that there exists a good interval $T \ni x$ of time n. Observe that all U_c are good intervals for U. Furthermore because of the ∂-condition we know that good intervals are nested: $f^i(T) \cap U = \emptyset$ for $i < n$. Because $f^n(\partial T) = \partial U_c$ the ∂-condition implies that $\mathrm{orb}(\partial T) \cap U = \emptyset$: $\partial T \subset \Lambda_U$ and $T = S$. Hence every component of the complement of Λ_U is a good interval. In particular, every connected component of the complement of $\Lambda_U \cup U$ is a good interval of time at least 1.

The orbits of points in Λ_U stay outside the neighborhood U of the critical points. The fact that the Lebesgue measure of such sets is zero is shown in [Mi] and [M]. ∎ (Lemma 3.4)

Lemma 3.4 should not be confused with Theorem A. This Lemma states that outside the neighborhood U the space is filled, measure theoretically, with good intervals. It says nothing about the good intervals in U. Lemma 3.2, however, states that U is filled densely with good intervals of time at least 1. The content of Theorem A is that U is also filled measure theoretically with good intervals if and only if the dynamical system does not have a zero-dimensional absorbing set.

COROLLARY 3.5: *Let U_n, $n = 1, 2, \dots$ be nice neighborhoods of $C(f)$ with* $\mathrm{mesh}(U_n) \to 0$. *If X is a forward invariant set with positive Lebesgue measure, then for every $n \geq 1$ there exists a component C_n of U_n such that*

$$\lim_{n\to\infty} \frac{|X \cap C_n|}{|C_n|} = 1.$$

Proof: Take a density point $x \in X$. We may assume that $x \notin \Lambda_{U_n}$ for all $n \geq 1$ (Λ_{U_n} is the set of measure zero obtained by applying Lemma 3.4 to the neighborhood U_n). Then we can find a good interval T_n for the component C_n of U_n with $x \in T_n$. Say $f^{k_n}: T_n \to C_n$. Observe that $|T_n| \to 0$.

Then

$$\begin{aligned}\lim_{n\to\infty} \frac{|X^c \cap C_n|}{|C_n|} &\leq \lim_{n\to\infty} \frac{|f^{k_n}(X^c \cap T_n)|}{|f^{k_n}(T_n)|} \\ &= \lim_{n\to\infty} \frac{|X^c \cap T_n|}{|T_n|} = 0.\end{aligned}$$

This finishes the proof. ∎ (Corollary 3.5)

An **ergodic component** of f is a forward and backward invariant set with minimal positive Lebesgue measure. Corollary 3.5 shows that there are at most as much ergodic components of f as there are critical points.

Given a neighborhood of a critical point it will in general not satisfy the nice property of Lemma 3.3. The next Lemma shows how we can deal with this problem.

LEMMA 3.6: *Let $V \subset \mathrm{int}(N)$ be an interval containing one critical point $c \in C_f$ and satisfying the ∂-condition. Let $K = \{x \in C_f \,|\, \mathrm{orb}(x) \cap V \neq \emptyset\}$. Then there exists a neighborhood $U = \bigcup_{d \in K} U_d$ of K with the following properties.*

(1) *$U_c = V$ and every component U_d of U contains only one critical point $d \in C_f$. Furthermore U satisfies the ∂-condition,*

(2) *there is a function $l: (0,1) \to \mathbb{R}$ with $l(y) \to 0$ if $y \to 0$ such that* $\mathrm{mesh}(U) \leq l(|V|)$,

(3) *the set* K *is partitioned, say* $K = \bigcup_{j \le s} K_j$ *so that*

$K_0 = \{c\}$,

for every $d \in K_j$, $j > 0$, *there exist an interval* $T_d \ni f(d)$, $n \ge 0$ *and some* $i < j$ *such that* $e \in K_i$ *such that* $f^n\colon T_d \to U_e$ *is monotone and onto and* U_d *is the connected component of* $f^{-1}(T)$ *containing* d,

(4) *if* $f^i(x) \notin U$ *for* $i < n$ *but* $f^n(x) \in U_d \subset U$, *there exists an interval* $T \ni x$ *such that* $f^n\colon T \to U_d$ *is monotone and onto.*

Proof: We construct U by induction. Assume we defined the objects:

(i) disjoint sets $K_0 = \{c\}, K_1, K_2, \ldots, K_s \subset K \subset C_f$,

(ii) neighborhoods $W_0 = V \subset W_1 \subset \cdots \subset W_s$ of the form $W_l = \bigcup_{i=0}^{l} \bigcup_{d \in K_i} U_d$ where U_d is the connected component of W_l containing d. In particular $W_0 = U_c = V$ and $W_l \supset \bigcup_{i=0}^{l} K_i$,

(iii) numbers $t_1, t_2, \ldots, t_l$ such that:

every point $d \in K_i$, $i > 0$, enters for the first time W_{i-1} after t_i steps, say $f^{t_i}(d) \in U_e$ with $e \in K_j, j < i$,

there exists an interval $T \ni v = f(d)$ such that $f^{t_i - 1}$ maps T monotonically onto U_e and U_d is the connected component of $f^{-1}(T)$ containing d,

(iv) W_i, $i \le s$ satisfies the ∂-condition.

Assume that $\bigcup_{l \le s} K_l \ne K$. We are going to define K_{s+1}, W_{s+1} and t_{s+1} according to the above properties. Take $x \in K \setminus \bigcup_{i \le s} K_i$ and let $t_{s+1}(x)$ be the first moment that the orbit of x enters W_s. This happens because $x \in K$ and $V \subset W_s$. Now let

$$t_{s+1} = \min\{t_{s+1}(x)\} \quad \text{and} \quad K_{s+1} = \{x \in K - \bigcup_{i \le s} K_i | t_{s+1}(x) = t_{s+1}\}.$$

To finish the construction we have to find the intervals $U_d = f^{-1}(T_d)$ for the points $d \in K_{s+1}$. Choose $d \in K_{s+1}$, say $f^{t_{s+1}}(d) \in U_e$ with $e \in K_j$, $j \le s$. Consider the maximal interval $M \ni v = f(d)$ on which $f^{t_{s+1}-1}$ is monotone and assume that the monotone image does not cover U_e. Because we assumed that f is part of an extension, the monotonicity is restricted by some critical point and not by a boundary point. There exist a critical point $e' \in C_f$ and a number $k < t_{s+1} - 1$ such that $e' \in \partial f^k(M)$ and $f^{t_{s+1}-1-k}((e', f^k(v)))$ is strictly contained in U_e. Observe that every point in W_s eventually enters V. So $\mathrm{orb}(e')$ intersects V: $e' \in K$. Because $f^{t_{s+1}-1-k}(e') \in U_e$ and $t_{s+1} - 1 - k < t_{s+1}$ we get $e' \in K_0 \cup K_1 \cup \cdots \cup K_s$. Hence $e' \in U_{e'} \subset W_s$. Because $k < t_{s+1} - 1$ we have $f^k(v) \notin U_{e'}$: $\partial U_{e'} \cap (e', f^k(v)) \neq \emptyset$. This implies that $\mathrm{orb}(\partial U_{e'}) \cap U_d \neq \emptyset$. This cannot be because W_s satisfies the ∂-condition. This contradiction implies $f^{t_{s+1}-1}(M) \supset U_e$. Now we can take the interval $T_d \ni v$ which is mapped by $f^{t_{s+1}-1}$ monotonically onto U_e, and we let U_d be the connected component of $f^{-1}(T_d)$ which contains d. This finishes the definition of W_{s+1}.

To finish the induction step we have to check that W_{s+1} satisfies the ∂-condition. To do so, take $y \in \partial W_{s+1}$ and assume by contradiction that for some $n \geq 0$ we have $f^n(y) \in W_{s+1}$, say $f^n(y) \in U_e$ with $e \in \bigcup_{j=0}^{s+1} K_j$. Because every point in W_{s+1} enters after some time W_s, we know that $y \in \partial U_d$ with $d \in K_{s+1}$. Because $f^{t_{s+1}}(y) \in \partial W_s$ and W_s satisfies the ∂-condition, we have $n < t_{s+1}$. Hence $f^n(d) \notin U_e$. So $\partial U_e \cap f^n(U_d) \neq \emptyset$ and $\mathrm{orb}(\partial U_e)$ intersects W_s, a contradiction.

This procedure will stop after finitely many steps: $\bigcup_{j \leq s} K_j = K$. Let $U = W_s$. Clearly U satisfies the ∂-condition. The Contraction Principle from [MMS] implies that $\mathrm{mesh}(U)$ goes to zero if $|V|$ goes to zero.

It remains to prove property (4). Take $x \in N$ and suppose that x enters U for the first time in $n \geq 0$ steps, say $f^n(x) \in U_d$. Let $M \ni x$ be the maximal interval on which f^n is monotone and suppose that $f^n(M)$ does not cover U_d. Because we assumed f to be part of an extension, the monotonicity is restricted by a critical point: there exist $e \in C_f$ and $i < n$ such that $e \in \partial f^i(M)$ and $f^{n-i}((e, f^i(x))$ is strictly contained in U_d. First observe that this implies $e \in K$. So $f^i(x) \notin U_e$ which implies that $\partial U_e \cap (e, f^i(x)) \neq \emptyset$. Hence $\mathrm{orb}(\partial U_e) \cap U \neq \emptyset$. This is impossible because U satisfies the ∂-condition. ∎ (Lemma 3.6)

4. The Markov property

As we will see in this section, every nice neighborhood of the critical points of a piecewise affine map defines an induced map. This induced map is strongly related to the ergodic theoretical behavior of the map. In particular the existence of absorbing Cantor sets is related to these induced maps. We will start to define these induced maps for the non-renormalizable piecewise affine map $f: N \to N$.

Fix a nice neighborhood $U \subset N$ of C_f with $\mathrm{mesh}(U)$ small enough to apply Lemma 3.2. Let $D \subset N$ be the union of all good intervals for U whose time is at least 1. From Lemma 3.2 we know that D is dense in N, and because good intervals are disjoint or nested, we get that every connected component of D is a good interval of time at least 1. This allows us to define the **Markov map**

$M: D \to U$ **(defined by** U**)** in the following way: for every connected component $T \subset D$ of D, with time n, set

$$M|T = f^n|T.$$

Observe that Markov maps are defined topologically.

Definition 4.1: A piecewise affine map $f: N \to N$ has the **Markov property** if there exists a nice neighborhood $U \subset N$ of the critical points such that its Markov map $M: D \to U$ is defined almost everywhere, i.e.,

$$|N - D| = 0.$$

The points in the set $B_0 = N - D$ are called **bad points**.

A closed set $A \subset N$ is called an **absorbing set** if

$$|\{x \in N | \omega(x) \subset A\}| > 0,$$

where $\omega(x)$ denotes the ω-limit set of $x \in N$.

THEOREM A: *A non-renormalizable piecewise affine map has the Markov property if and only if it does not have zero-dimensional absorbing sets.*

The next two lemmas are needed as preparation for the proof of Theorem A. The first one gives a description of the limit behavior of bad points. The second one is technical but will be used to prove the ergodicity of non-renormalizable maps. It also enables us to define special Markov maps whose image is just one interval. These Markov maps play a crucial role in the description of conjugacy classes in Section 5.

To describe the limit behavior of bad points, we need some preparation. Fix a nice neighborhood $U \subset N$ of C_f and consider the Markov map $M: D \to U$. Let $B_0 = N - D$ be the closed zero-dimensional set of bad points. In general this set will not be invariant. The main step in the proof of Theorem A is to find an "almost invariant" set $\hat{B} \supset B_0$.

We start to define the **depth** $d(T)$ of a good interval T as the number of good intervals which contain strictly T. Clearly every component of D has depth equal to zero. If the critical value $f(c)$ is contained in a good interval of depth d, denote this interval by $T_d(c)$. A critical point which has infinitely many T_d's is said to be of **infinite type**. Otherwise, it is of **finite type**.

We also need to define the **pull back** of bad points along the orbit of a good interval. Let $T \subset N$ be a good interval, say $f^n: T \to U_c$. Define the **tubes**

$$P_T = \bigcup_{i=0}^{n} \overline{f^{-i}(B_0 \cap U_c) \cap f^{n-i}(T)}.$$

Now the **extended set of bad points** is defined to be

$$\hat{B} = \bigcup_{n \geq 0} B_n,$$

where

$$B_n = B_0 \cup \bigcup_{d=0}^{n} \bigcup_{c \in C_f} P_{T_d(c)}.$$

Clearly every B_n is a closed zero-dimensional set.

LEMMA 4.2: *The extended set of bad points contains* B_0 *and satisfies* $f(\hat{B} - C_f) \subset \hat{B}$. *For almost every* $x \in B_0$, *there exists* $n \geq 0$ *with*

$$\text{orb}(x) \subset B_n.$$

Proof: Assume that f is part of an extension. First we will show the near invariance property of $\hat{B}$. Because the tubes P_T are invariant, it suffices to show that $f(B_0 - C_f) \subset \hat{B}$. Let $x \in B_0 - C_f$. If $f(x) \notin B_0$, there exist $c \in C_f$ and $d \geq 1$ with $f(x) \in T_d(c)$. Because $x \neq c$, we can take d maximal with these properties. Now $f(x) \notin \hat{B}$ implies $f(x) \notin P_{T_d(c)}$. So there exists some good interval $T \subset T_d(c)$ with $f(x) \in T$, and because d was taken to be maximal, we have $f(c) \notin T$. Then $f^{-1}(T)$ contains a good interval around $x \in B_0$, a contradiction.

To prove the second statement take some $x \in B_0$. We may assume that x is not a preimage of any critical point. Now assume that $\text{orb}(x) \not\subset B_n$ for all $n \geq 1$. We are going to show that x is not a density point of B_0.

Given $n \geq 1$ the orbit of x leaves B_n after some steps. There is only one way to leave B_n. The point reurns to some U_c, $c \in C_f$, and falls into some $V_d(c)$ with $d \geq n+1$.

There are two observations to be made:

$|V_d(c)| \to 0$ if $d \to \infty$,

$V_d(c)$ satisfies the ∂-condition.

The first observation is clear. Assume that $V_d(c)$ does not satisfy the ∂-condition. Then there is a monotone image of $T_d(c)$ intersecting $V_d(c)$. This image can not be contained in $V_d(c)$ otherwise there would be a trap. Hence this image intersects the boundary of $V_d(c)$. This would imply that some monotone image of $T_d(c)$ and $T_d(c)$ intersect but the two good intervals are not nested. Contradiction.

This means that we can apply Lemma 3.6 and get nice extensions $W_d(c)$ of $V_d(c)$ with $\text{mesh}(W_d(c)) \to 0$ if $d \to \infty$.

It only remains to show that for some $\epsilon > 0$,

$$\frac{|D \cap W|}{|W|} \geq \epsilon$$

for every component $W \subset W_d(c)$ and every $c \in C_f$ and $d \geq 1$: we can push back this definite amount of good intervals into a very small neighborhood of x by using Lemma 3.6 again, showing that x is not a density point of B_0. Density points could not go too deep in $\hat{B}$, and the Lemma will be proved.

Let $K \subset \overline{K} \subset I$ be two open intervals. The set $A = I - K$ is called a **boundary piece** if $|\{T \subset A | T \text{ is a good interval }\}| = |A|$. The first step is to show that every U_c has a boundary piece. Fix $c \in C_f$ and consider the sequence of intervals $Q_1 = f(U_c), Q_2, \dots$ with the following inductive definition: If $Q_i \subset T$ where T is a good interval for U_d, then $Q_{i+1} = f(U_d)$. Otherwise the sequence stops. If this sequence is longer than the number of critical points, then at least one critical point is visited at least twice, and there is a trap. Hence this sequence is finite. Say $Q_s = f(U_d)$ is not subset of $T_1(d)$. Now apply Lemma 3.4 and we see that U_d has a boundary piece. Considering the sequence $Q_s, \dots, Q_2, Q_1$, we can pull back parts of this piece and we will find a boundary piece in U_c.

The second step is to make definite boundary pieces in the $V_n(d)$. This is easy because we can pull back one of the above boundary pieces into $T_n(c)$, giving rise to definite boundary pieces in $T_n(d)$. One step more and we will find the definite boundary pieces in $V_n(c)$.

Lemma 3.6 describes how the different components of $W_n(d)$ are related: they form a tree. Using this description we can pull back the definite boundary pieces in $V_n(c)$ into definite boundary pieces of the components of $W_n(c)$.

Observe that the only non-bounded part of the construction takes place during the transport of the boundary pieces in U_c to the $T_n(d)$. This transport is affine so that the proportion of space occupied by boundary pieces is preserved, as well as the fact that these boundary pieces are filled by good intervals. ∎ (Lemma 4.2)

LEMMA 4.3: *Let $U = \bigcup_{c \in C_f} U_c \supset C_f$ be a nice neighborhood with* mesh(U) *small enough. The set D_∞ consists of all points $x \in N$ which are contained in infinitely may good intervals:*

$$x \in \cdots \subset T_3(x) \subset T_2(x) \subset T_1(x)$$

with $f^{t_i(x)}\colon T_i(x) \to U_{c_i(x)}$ and $t_i(x) \to \infty$.

For every critical point $c \in C_f$ and for almost all $x \in D_\infty$ there are infinitely $T_i(x)$ with $f^{t_i(x)}\colon T_i(x) \to U_c$.

In particular, if $B \subset U_c$ with $|B_0 \cap U_c - B| = 0$ and $|B| > 0$, then almost every $x \in D_\infty$ hits B after some time.

Proof: Fix $c \in C_f$. Lemma 3.2 implies that every U_d contains a good interval for U_c. Let $B \subset U$ be the union of those good intervals and

$$X_B = \{x \in D_\infty |\, \mathrm{orb}(x) \cap B = \emptyset\}.$$

Take $x \in D_\infty$ and consider the sequence $x \in \cdots \subset T_2(x) \subset T_1(x)$ of good intervals with times $t_i(x) \to \infty$. Let $B_i = T_i \cap f^{-t_i}(B)$. Clearly $B_i \cap X_B = \emptyset$. Because $f^{t_i}|T_i$ is affine we get

$$\frac{|B_i|}{|T_i|} = \frac{|B \cap U_{c_i(x)}|}{|U_{c_i(x)}|} \geq \min_{c \in C_f} \frac{|B \cap U_c|}{|U_c|} \geq \epsilon > 0$$

for all $i \geq 1$. Because there are no wandering intervals and no periodic attractors we have $|T_i(x)| \to 0$. Hence x is not a density point of X_B and $|X_B| = 0$. Now D_∞ is covered up to a set of measure zero by good intervals for U_c. From this we get directly that almost every point in D_∞ is contained in infinitely many good intervals for U_c.

Take a set $B \subset U_c$ with $|B| > 0$ and $|B_0 \cap U_c - B| = 0$. Let $X_B = \{x \in D_\infty |\, \mathrm{orb}(x) \cap B = \emptyset\}$. As above we can show that $|X_B| = 0$.
∎ (Lemma 4.3)

Instead of proving Theorem A, we will prove the following stronger proposition describing more precisely the ergodic theoretical behavior. A map $f\colon N \to N$ is called **ergodic** if it does not have two disjoint ergodic components. It is called **conservative** if almost every point hits after some time an arbitrarily given set $X \subset N$ with positive Lebesgue measure.

PROPOSITION 4.4: *Let f be a non-renormalizable piecewise affine map.*

If f has the Markov property, f is ergodic and conservative. In particular, the orbit of almost every point is dense in N.

If f has a Markov map $M: D \to U$ with $|N - D| > 0$, then there exists $s \geq 0$ such that for almost all $x \in N$

$$\mathrm{orb}(f^{n_x}(x)) \subset B_s$$

for $n_x \geq 0$ big enough. In particular

$$|N - \{x \in N | \omega(x) \subset B_s\}| = 0,$$

so that B_s is a zero-dimensional absorbing set, absorbing in fact almost every orbit.

Proof: Let f be a piecewise affine map having the Markov property. The Markov property implies that $|N - D_\infty| = 0$. Now let $X \subset N$ be an invariant set of positive Lebesgue measure. Take $c \in C_f$ and a density point $x \in X \cap D_\infty$ of X. Now consider only the intervals $T_i(x)$ from Lemma 4.3 which are good for U_c and observe that $\frac{|X \cap T_i|}{|T_i|} \to 1$. Because X is invariant we conclude that $\frac{|X \cap U_c|}{|U_c|} = 1$. Conclusion: we cannot have two disjoint invariant sets of positive measure: the map f is ergodic.

To prove the conservativity of f, take a set $A \subset N$ with positive Lebesgue measure. From Proposition 2.1 we know that there is some $J \subset U_c$ with positive Lebesgue measure and some number $n \geq 0$ such that $f^n(J) \subset A$. Now apply Lemma 4.3 to $B = J$. Almost every point enters J after some time, hence also enters A a little bit later.

Consider next a piecewise affine map f which does not have the Markov property. Then there exists a Markov map $M: D \to U$ with $|B_0| > 0$. From Lemma 3.4 we get $|B_0 - U| = 0$. Hence there exists a $c \in C_f$ with $|U_c \cap B_0| > 0$. As a direct consequence of Lemma 4.3, we get $|D_\infty| = 0$: almost every point hits B_0 after some time. The limit behavior of orbits is guided by the behavior of points in B_0. This behavior is described by Lemma 4.2, giving rise to the following candidates for the ergodic components:

$$E_n = \bigcup_{i \in \mathbb{Z}} f^i(E'_n),$$

where

$$E'_n = \{x \in B_0 |\, \text{orb}(x) \subset B_n \text{ and } \text{orb}(x) \not\subset B_{n-1}\}.$$

These sets E_n are pairwise disjoint, forward and backward invariant sets. Furthermore, by Lemma 4.2, we get $|\bigcup_{n\geq 0} E'_n| = |B_0|$, and as a consequence of Lemma 4.3, $|N - \bigcup_{n\geq 1} E_n| = 0$. Now Corollary 3.5 implies that there are only finitely many E_n with $|E_n| > 0$. Hence for some $s \geq 0$

$$|N - \bigcup_{n\leq s} E_n| = 0.$$

This means that the limit behavior takes place in B_s:

$$|N - \{x \in N | \omega(x) \subset B_s\}| = 0,$$

so that B_s is a zero-dimensional set absorbing almost all points in N.
∎ (Proposition 4.4)

Remark: Proposition 4.4 implies that if a map has the Markov property, then all its Markov maps are defined almost everywhere.

THEOREM B: *An eventually expanding non-renormalizable piecewise affine map has the Markov property.*

Proof: In [LY] it was proved that an eventually expanding piecewise affine map has an absolutely continuous invariant probability measure, and that furthermore, the density of this measure has bounded variation.

Now assume that there is an eventually expanding non-renormalizable piecewise affine map not having the Markov property. Given a Markov map $M: D \to U$, there is some $s \geq 0$ such that the orbit of almost every point enters the closed set B_s after some time. Hence every ergodic component of the invariant probability measure is supported on B_s. In fact the whole measure is supported on B_s. Now B_s is zero dimensional and closed, and such sets cannot support a non-zero density of bounded variation. This yields a contradiction. ∎ (Theorem B)

5. Conjugacy classes

In this section we are going to consider families of piecewise affine maps and show that every conjugacy class is contained in a submanifold of codimension 1 in the space of such maps.

Let $\mathcal{F}_d$ be the family of d-modal piecewise affine maps. The subfamily of eventually expanding piecewise affine maps is denoted by $\mathcal{E}_d \subset \mathcal{F}_d$. We consider $\mathcal{E}_d$ as a submanifold of $\mathbb{R}^{2d+2}$. We study the conjugacy question inside the class $\mathcal{E}_d$. The conjugacy class of a map $f \in \mathcal{E}_d$ is denoted by $[f] \subset \mathcal{E}_d$.

To describe the conjugacy classes we need the notion of essential branches and slopes. The i^{th} branch is called **essential** if a minimal cycle intersects the interior of its domain. The slope $Df(i)$ is then also called **essential.** Let B_f be the collection of the essential branches. Observe that B_f is defined topologically. In general one can change non-essential slopes of a map without changing its topological type: examples are easily provided.

THEOREM C: *The conjugacy class $[f] \subset \mathcal{E}_d$ is contained in a codimension 1 submanifold of $\mathcal{E}_d$. In particular, if $g \in [f]$ and its essential slopes are at least as big as the corresponding essential slopes of f then they are in fact equal, i.e.,*

$$\{|Dg(i)| \geq |Df(i)|, i \in B_f\} \Rightarrow \{Df(i) = Dg(i) \quad \text{for } i \in B_f\}.$$

The proof of this Theorem is based on the Multiple Covering Map principle. We will not work in $\mathcal{E}_d$, but in the space of inverses of slopes. To go back to $\mathcal{E}_d$ we use the submersion

$$\pi: \mathcal{E}_d \to D_d = (0, \infty)^{d+1}$$

defined by $\pi(f)(i) = (|Df(i)|)^{-1}$.

The basic step in proving Theorem C is the definition of an induced map. This will allow us to define topologically a multiple covering map for every $f \in \mathcal{E}_d$.

Choose a map $f \in \mathcal{E}_d$ and consider the minimal cycle corresponding to a non-renormalizable renormalization (N, n). Let $B \subset B_f$ be the collection of essential branches which are used by the cycle $\bigcup f^i(N)$ and let $D = (0, \infty)^B$. The natural projection from D_d into D is denoted by $p: D_d \to D$. Let $g = f^n|N$ and take a nice neighborhood $U \subset N$ of C_g. Choose $c \in C_g$ and consider the union $G \subset U_c$ of all good intervals $T \subset U_c$ for U_c of positive time. As in the definition of Markov maps we get an induced map $T: G \to U_c$. Observe that the range being induced to is the same, up to a nowhere dense set, as the domain on which inducing occurs.

Since we started with an eventually expanding map f, the map g is also eventually expanding. Hence it has the Markov property. Now by applying Lemma 4.3 we get that

$$|U_c - G| = 0,$$

i.e., T is a multiple covering map.

Before applying the Multiple Covering Map principle, we need some definitions. Let B_T be the collection of branches of T. For every $I \in B_T$ there is a unique $t_I \geq 1$ such that $T|I = f^{t_I}|I$. The only thing left over is to count how many times the orbits of those branches of T use the branches of f. Let $I \in B_T$ and $i \in B$ and define $t_I(i) = \#\{j \leq t_I - 1 | f^j(I) \subset i\}$.

Let $p \circ \pi(f) = (y_1, \ldots, y_b) \in D$. Then the Multiple Covering Map principle tells us

$$\sum_{I \in B_T} \prod_{i \in B} y_i^{t_I(i)} = 1.$$

Now observe that the objects U_c, T, B, B_T and $t_I(i)$ are all topologically defined. So if we define a real analytic function $\psi: D \to \mathbb{R}$ by

$$\psi(x) = \sum_{I \in B_T} \prod_{i \in B} x_i^{t_I(i)}$$

then $\psi \circ p \circ \pi(f') = 1$ for all $f' \in [f]$. So

$$p \circ \pi([f]) \subset \psi^{-1}(1).$$

The sequence

$$\mathcal{E}_d \xrightarrow{\pi} D_d \xrightarrow{p} D \xrightarrow{\psi} \mathbb{R} \ni 1$$

indicates how to prove Theorem C: we have to show that 1 is a regular value of $\psi \circ p \circ \pi$. In particular we will show that the gradient of ψ has only positive entries. This will also imply the second statement of Theorem C.

Proof of Theorem C: Take $f \in \mathcal{E}_d$ and let $F = p \circ \pi([f]) \subset D$. Let $W \subset D$ be the interior of the domain of convergence of ψ. Then for $x \in W$ the gradient of ψ is defined and its components are given by

$$\frac{\partial \psi}{\partial x_i}(x) = \sum_{I \in B_T(i)} t_I(i) x_i^{t_I(i)-1} \prod_{j \neq i} x_j^{t_I(j)} > 0,$$

where $B_T(i) = \{I \in B_T | t_I(i) > 0\}$, i.e. consists of those branches whose orbits pass trough the branch $i \in B$. Such branches exist because $g|N$ is non-renormalizable: the g-orbit of U_c is dense in N, hence the f-orbit of U_c is dense in the minimal cycle.

The fact that all components of the gradient of ψ are positive implies that 1 is a regular value of ψ. So $\psi^{-1}(1) \cap W$ is an analytic codimension 1 submanifold of D.

CASE 1: $F \subset W$. Then F is part of the analytic codimension 1 submanifold $\psi^{-1}(1) \cap W$, hence $[f]$ is contained in the analytic codimension 1 submanifold $(\psi \circ p \circ \pi)^{-1}(1)$. Furthermore, because ψ has a positive gradient, we will leave $\psi^{-1}(1)$, and so F, by increasing some essential slopes. Theorem C is proved.

CASE 2: $F \not\subset W$. In this case we will glue together a piece of the analytic manifold $\psi^{-1}(1) \cap W$ with a piece of the boundary of W to obtain a continuous codimension 1 submanifold of D containing F. To do so we have to study the boundary of W.

Let $x = (x_i) \in D$ and consider the cubes

$$Q_x^- = \{(y_i) \in D | y \neq x \text{ and } 0 \leq y_i \leq x_i\}$$

and

$$Q_x^+ = \{(y_i) \in D | y \neq x \text{ and } y_i \geq x_i\}.$$

Because the coefficients of the series ψ are all non-negative we get the **cube property**

$$\psi(y) < \psi(x) < \psi(z)$$

for all $y \in Q_x^-$ and $z \in Q_x^+$.

We will use polar coordinates on D. Let S be the set of all Euclidean unit vectors in D. Because $F \not\subset W$ there is a $\theta_0 \in S$ such that the ray $\{\theta_0\} \times (0, \infty) \not\subset W$. Using the cube property of ψ we conclude that all rays $\{\theta\} \times (0, \infty)$ are not contained in W. Again, because the coefficients of the series ψ are all non-negative ψ is strictly increasing along rays. Now we can describe the boundary of W as the graph of a function b: $S \to \mathbb{R}$. The cube property of ψ implies easily that b is continuous (Lipschitz on compact sets of S). We conclude that the boundary of W is a continuous codimension 1 submanifold of D.

Every ray contains at most one point of $\psi^{-1}(1)$ because ψ is strictly increasing along rays. Let $X = \{\theta \in S | \{\theta\} \times (0, \infty) \cap \psi^{-1}(1) \cap W \neq \emptyset\}$. Because the gradient of ψ has only positive entries we get that $X \subset S$ is open and there exists an analytic function m: $X \to \mathbb{R}$ such that the graph of m is the analytic manifold $\psi^{-1}(1) \cap W$.

Now we glue those two graphs together to get a continuous codimension 1 submanifold of D containing F. To do so observe that for a sequence $x_n \in X$ tending to $x \in \partial X$ we have that $(x_n, m(x_n))$ tends to the boundary of W. So the following function is continuous:

$$k(x) = \begin{cases} m(x), & \text{if } x \in X \\ b(x), & \text{if } x \notin X \end{cases}$$

and its graph in D is a continuous codimension 1 submanifold of D containing F.

Let g be the map obtained by increasing an essential slope of f, say $p(\pi(f)) = x \in D$ and $p(\pi(g)) = y \in D$. Then $y \in Q_x^-$. The cube property says that we leave $\psi^{-1}(1)$ and hence the conjugacy class of f. ∎ (Theorem C)

Markov maps can be used to define Markov extensions (the original map is a factor of the extension). Those extensions turned out to be very useful for studying the ergodic theoretical behavior, especially the existence of invariant measures. This is because extensions always have an absolutely continuous invariant measure. Invariant measures for the original map could be constructed by trying to project the measure of the extension.

The question whether every continuous measure can be obtained by projecting is not settled. Moreover, it is strongly related to the question whether case 2 in the proof of Theorem C actually occurs. Observe that the measure of the Markov extension of a map f can be projected if and only if the gradient of ψ is finite in the corresponding point in D. So showing that case 2 does not happen would also solve the projection question. Under certain combinatorial conditions it is possible to show the

CONJECTURES:

(1) *Conjugacy classes are contained in analytic submanifolds,*

(2) *The invariant measure of the Markov extension can be projected.*

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